Wednesday, December 2, 2015

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Problem 1

Problem. Verify the formula $\frac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1)$.

Solution. Expand each of the factorials and then cancel common factors, namely, the factors from $n-2$ down to 1.

$$\begin{aligned}\frac{(n+1)!}{(n-2)!} &= \frac{(n+1)(n)(n-1)(n-2)\cdots(3)(2)(1)}{(n-2)\cdots(3)(2)(1)}\\ &= (n+1)(n)(n-1).\end{aligned}$$

Problem 5

Problem. Match the series $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$ with the graph of its sequence of partial sums.

Solution. Calculate the first few partial sums.

$$\begin{aligned}S_1 &= \frac{3}{4},\\ S_2 &= S_1 + (2)\left(\frac{3}{4}\right)^2\\ &= \frac{3}{4} + \frac{9}{8}\\ &= \frac{15}{8}\\ &= 1.875,\\ S_3 &= S_2 + (3)\left(\frac{3}{4}\right)^3\\ &= \frac{15}{8} + \frac{81}{64}\\ &= \frac{201}{64}\\ &\approx 3.14.\end{aligned}$$

This must be (d).

Problem 6

Problem. Match the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right)$ with the graph of its sequence of partial sums.

Solution. Calculate the first few partial sums.

$$\begin{aligned}
S_1 &= \frac{3}{4}, \\
S_2 &= S_1 + \left(\frac{3}{4}\right)^2 \left(\frac{1}{2}\right) \\
&= \frac{3}{4} + \frac{9}{32} \\
&= \frac{33}{32} \\
&\approx 1.03, \\
S_3 &= S_2 + \left(\frac{3}{4}\right)^3 \left(\frac{1}{6}\right) \\
&= \frac{15}{8} + \frac{9}{128} \\
&= \frac{141}{128} \\
&\approx 1.101.
\end{aligned}$$

This must be (c).

Problem 8

Problem.

Match the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$ with the graph of its sequence of partial sums.

Solution. Calculate the first few partial sums.

$$
\begin{aligned}
S_1 &= \frac{4}{2} = 2, \\
S_2 &= S_1 - \frac{4}{24} \\
&= \frac{11}{6} \\
&\approx 1.83, \\
S_3 &= S_2 + \frac{4}{720} \\
&\approx 1.84.
\end{aligned}
$$

This must be (b).

Problem 9

Problem. Match the series $\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3} \right)^n$ with the graph of its sequence of partial sums.

Solution. Calculate the first few partial sums.

$$
\begin{aligned}
S_1 &= \frac{4}{2} = 2, \\
S_2 &= S_1 + \left(\frac{8}{7} \right)^2 \\
&\approx 3.31, \\
S_3 &= S_2 + \left(\frac{12}{12} \right)^3 \\
&\approx 4.31.
\end{aligned}
$$

This must be (a).

Problem 15

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{n!}{3^n}$.

Solution.

Problem 16

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} \\
&= \lim_{n\to\infty} \left(\frac{(n+1)!}{3^{n+1}} \right) \left(\frac{3^n}{n!} \right) \\
&= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \right) \left(\frac{3^n}{3^{n+1}} \right) \\
&= \lim_{n\to\infty} \frac{n+1}{3} \\
&= \infty.
\end{aligned}$$

Therefore, the series diverges.

Problem 21

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{n^3}{n3^n}$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\left(\frac{(n+1)^3}{(n+1)3^{n+1}} \right)}{\left(\frac{n^3}{n3^n} \right)} \\
&= \lim_{n\to\infty} \left(\frac{(n+1)^3}{n^3} \right) \left(\frac{n3^n}{(n+1)3^{n+1}} \right) \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^3 \left(\frac{n}{3(n+1)} \right) \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \left(\frac{1}{3} \right) \\
&= \frac{1}{3}.
\end{aligned}$$

Therefore, the series converges.

Problem 22

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{(1-)^{n+1}(n+2)}{n(n+1)}$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\left(\frac{n+3}{(n+1)(n+2)}\right)}{\left(\frac{n+2}{n(n+1)}\right)} \\
&= \lim_{n\to\infty} \left(\frac{n+3}{(n+1)(n+2)}\right)\left(\frac{n(n+1)}{n+2}\right) \\
&= \lim_{n\to\infty} \frac{n(n+3)}{(n+2)^2} \\
&= 1.
\end{aligned}$$

The Ratio Test is inconclusive. Instead, use the Alternating Series Test. The terms are alternating and it is clear that $\frac{n+2}{n(n+1)} \to 0$ as $n \to \infty$. We need only check that $a_{n+1} \le a_n$.

$$\begin{aligned}
\frac{n+3}{(n+1)(n+2)} &\le \frac{n+2}{n(n+1)}, \\
(n+3)(n)(n+1) &\le (n+1)(n+2)^2 \\
(n+3)(n) &\le (n+2)^2 \\
n^2+3n &\le n^2+4n+4 \\
0 &\le n+4,
\end{aligned}$$

which is clearly true for all $n \ge 1$ and the steps are logically reversible. Therefore, the series converges.

Note that the Ratio Test will *always* fail when the terms of the series are rational functions of n.

Problem 30

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$.

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\left(\frac{((n+1)!)^2}{(3(n+1))!}\right)}{\left(\frac{(n!)^2}{(3n)!}\right)} \\
&= \lim_{n\to\infty}\left(\frac{((n+1)!)^2}{(3(n+1))!}\right)\left(\frac{(3n)!}{(n!)^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{((n+1)!)^2}{(n!)^2}\right)\left(\frac{(3n)!}{(3(n+1))!}\right) \\
&= \lim_{n\to\infty}(n+1)^2\left(\frac{1}{(3n+1)(3n+2)(3n+3)}\right) \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} \\
&= 0.
\end{aligned}
$$

Therefore, the series converges.

Solution.

Problem 33

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n+1)}$.

Solution.

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\left(\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+3)}\right)}{\left(\frac{n!}{1\cdot 3\cdot 5\cdots(2n+1)}\right)} \\
&= \lim_{n\to\infty}\left(\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+3)}\right)\left(\frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!}\right) \\
&= \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)\left(\frac{1\cdot 3\cdot 5\cdots(2n+1)}{1\cdot 3\cdot 5\cdots(2n+3)}\right) \\
&= \lim_{n\to\infty}(n+1)\left(\frac{1}{2n+3}\right) \\
&= \lim_{n\to\infty}\frac{n+1}{2n+3} \\
&= \frac{1}{2}.
\end{aligned}
$$

Therefore, the series converges.

Problem 34

Problem. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n [2 \cdot 4 \cdot 6 \cdots (2n)]}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\left(\frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}\right)}{\left(\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)}\right)} \\
&= \lim_{n\to\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)}{2 \cdot 5 \cdot 8 \cdots (3n-1)(3n+2)} \right) \left(\frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right) \\
&= \lim_{n\to\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right) \left(\frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)(3n+2)} \right) \\
&= \lim_{n\to\infty} \frac{2n+2}{3n+2} \\
&= \frac{2}{3}.
\end{aligned}$$

Therefore, the series converges.